On the Classification of Nilpotent Lie Algebras

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Abstract. We establish a bijective correspondence between all central extensions of fixed Lie algebra $\mathfrak{g}$ by $\mathbb{R}^k$ (with k-dimensional centers), and certain orbits in the set of all k-dimensional subspaces in the second cohomology group $H^2(\mathfrak{g},\mathbb{R})$, under the canonical action of $\mathrm{Aut}(\mathfrak{g})$. As an application we construct all six-dimensional nilpotent Lie algebras over $\mathbb{R}$ (finitely many). We also show that there are infinitely many mutually non-isomorphic two-step real Lie algebras of dimension nine.

1. Introduction.

In this article we develop a method of constructing all nilpotent Lie algebras of dimension n given those algebras of dimension $< n$, and their automorphism groups. Roughly speaking, we establish a bijective correspondence between all central extensions of a fixed Lie algebra $\mathfrak{g}$ by $\mathbb{R}^k$ with k-dimensional centers, and certain orbits in the set of all k-dimensional subspaces in the second cohomology group $H^2(\mathfrak{g},\mathbb{R})$, under the canonical action of $\mathrm{Aut}(\mathfrak{g})$. For natural reasons we are working in the space S of all bilinear forms on $\mathfrak{g}$ with values in $\mathbb{R}^k$ satisfying the Jacobi identity, rather than in $H^2(\mathfrak{g},\mathbb{R}^k)$. If $B \in S$, the action of $\mathrm{Aut}(\mathfrak{g})$ on B is simply given by

$$(\alpha, B) \to B^\alpha, \text{ where } B^\alpha(X,Y) = B(\alpha X,\alpha Y);\ X,Y \in \mathfrak{g},\ \alpha \in \mathrm{Aut}(\mathfrak{g}).$$

As an application of our procedure, we find all nilpotent Lie algebras over $\mathbb{R}$ of dimension 6 (finitely many). Recall that Dixmier has classified all such algebras of dimension ≤ 5, [1]. We will also show that there are infinitely many mutually non-isomorphic real two-step Lie algebras of dimension 9 with center of dimension 3. Even if our method is straight forward, the computations of $\mathrm{Aut}(\mathcal{G})$-orbits in $H^2(\mathcal{G},\mathbb{R}^k)$ are in general far from easy to carry out, and at the moment a complete classification of nilpotent Lie algebras seems to be out of reach.

2. Central extensions and automorphisms.

2.1. Let $\mathcal{G}$ be a Lie algebra over $\mathbb{R}$, $B : \mathcal{G} \times \mathcal{G} \to \mathbb{R}^k$ a skew symmetric bilinear form satisfying the Jacobi identity

$$(2.1) \qquad B([X,Y],Z) + B([Z,X],Y) + B([Y,Z],X) = 0, \quad \text{all } X,Y,Z \in \mathcal{G}.$$

Such forms are said to be closed. If B is a closed form on $\mathcal{G}$ we construct a Lie algebra on $\mathcal{G} \oplus \mathbb{R}^k$, letting $\left[\begin{pmatrix} X \\ u \end{pmatrix}, \begin{pmatrix} Y \\ v \end{pmatrix}\right] = \begin{pmatrix} [X,Y] \\ B(X,Y) \end{pmatrix}$; $X,Y \in \mathcal{G}$, $u,v \in \mathbb{R}^k$. Denote this Lie algebra by $\mathcal{G}(B)$.

Let now $\tilde{\mathcal{G}}$ be a Lie algebra with center $\tilde{\mathfrak{z}}$ and $\dim \tilde{\mathfrak{z}} = k$, $\gamma : \tilde{\mathcal{G}} \to \mathbb{R}^k$ be linear and such that $\gamma(\tilde{\mathfrak{z}}) = \mathbb{R}^k$. We put $\mathcal{G} = \tilde{\mathcal{G}}/\tilde{\mathfrak{z}}$ and get an isomorphism $\tilde{\mathcal{G}} \cong \mathcal{G} \oplus \mathbb{R}^k$ where $\tilde{X} \longleftrightarrow \binom{Y}{u}$, $\gamma(\tilde{X}) = u$, and $Y = \tilde{X} + \tilde{\mathfrak{z}} \in \tilde{\mathcal{G}}/\tilde{\mathfrak{z}} = \mathcal{G}$. We put $B = \gamma \circ [\cdot,\cdot]$, that is

$$B(X,Y) = \gamma[X',Y']; \text{ where } X' + \tilde{\mathfrak{z}} = X, \; Y' + \tilde{\mathfrak{z}} = Y.$$

This shows $\tilde{\mathcal{G}}$ and $\mathcal{G}(B)$ are isomorphic. Hence each Lie algebra with center of dimension k is of the form $\mathcal{G}(B)$ where $B : \mathcal{G} \times \mathcal{G} \to \mathbb{R}^k$.

2.2. Let $B : \mathcal{G} \times \mathcal{G} \to \mathbb{R}^k$ be a closed form. Then the center $\tilde{\mathfrak{z}}$ of $\tilde{\mathcal{G}} = \mathcal{G}(B)$ is equal to

$$\tilde{\mathfrak{z}} = \mathbb{R}^k \oplus (\mathfrak{z} \cap \mathcal{S}_B), \quad \text{where} \quad \mathcal{S}_B = \{X \in \mathcal{G} : B(X, \mathcal{G}) = (0)\}$$

2.3. Given two such forms $B_1, B_2 : \mathcal{G} \times \mathcal{G} \to \mathbb{R}^k$, and assume the extended algebras $\mathcal{G}(B_1)$ and $\mathcal{G}(B_2)$ are isomorphic and that their centers $\tilde{\mathfrak{z}}_1$ and $\tilde{\mathfrak{z}}_2$ both are equal to $\mathbb{R}^k$. Let $\alpha : \mathcal{G}(B_1) \to \mathcal{G}(B_2)$ be an isomorphism. Dividing with the common center $\mathbb{R}^k$ we obtain an automorphism $\alpha_0 : \mathcal{G} \to \mathcal{G}$. Let us fix a basis $\{e_1, \ldots, e_n\}$ for $\mathcal{G}$, and supplement it with a basis for $\mathbb{R}^k$ to get a basis $\mathcal{E}$ for $\mathcal{G} \oplus \mathbb{R}^k$. We may realize α as a matrix relative to $\mathcal{E}$:

$$(2.2) \qquad \alpha = \left(\begin{array}{c|c} \alpha_0 & 0 \\ \hline \varphi & \psi \end{array}\right) ; \ \alpha_0 \in \mathrm{Aut}(\mathcal{G}),\ \psi = \alpha|_{\tilde{\mathfrak{z}}} \in GL(k), \text{ and } \varphi \in \mathrm{Hom}(\mathcal{G}, \mathbb{R}^k).$$

Now α preserves the brackets, and writing $[\cdot,\cdot]_i$ for the products in $\mathcal{G}(B_i)$, $i = 1,2$, and $[\cdot,\cdot]$ for the product in $\mathcal{G}$, we have

$$\alpha\left[\begin{pmatrix}X\\0\end{pmatrix}, \begin{pmatrix}Y\\0\end{pmatrix}\right]_1 = \left[\alpha\begin{pmatrix}X\\0\end{pmatrix}, \alpha\begin{pmatrix}Y\\0\end{pmatrix}\right]_2, \quad X, Y \in \mathcal{G},$$

where $\alpha\begin{pmatrix}X\\0\end{pmatrix} = \begin{pmatrix}\alpha_0 & X\\ \varphi & X\end{pmatrix}$, and hence

$$\left[\alpha\begin{pmatrix}X\\0\end{pmatrix}, \alpha\begin{pmatrix}Y\\0\end{pmatrix}\right]_2 = \left[\begin{pmatrix}\alpha_0 & X\\ \varphi & X\end{pmatrix}, \begin{pmatrix}\alpha_0 & Y\\ \varphi & Y\end{pmatrix}\right]_2 = \begin{pmatrix}[\alpha_0 X, \alpha_0 Y]\\ B_2(\alpha_0 X, \alpha_0 Y)\end{pmatrix}$$

and

$$\alpha\left[\begin{pmatrix}X\\0\end{pmatrix}, \begin{pmatrix}Y\\0\end{pmatrix}\right]_1 = \alpha\begin{pmatrix}[X,Y]\\ B_1(X,Y)\end{pmatrix} = \begin{pmatrix}\alpha_0[X,Y]\\ \varphi[X,Y] + \psi B_1(X,Y)\end{pmatrix}.$$

Hence

$$(2.3) \qquad B_2(\alpha_0 X, \alpha_0 Y) = \varphi[X,Y] + \psi B_1(X,Y), \quad \text{all} \quad X, Y \in \mathcal{G}.$$

In case $B_1 = B_2 = B$, we get the following description of the automorphism group $\mathrm{Aut}(\mathcal{G}(B))$.

2.4. Proposition. Let B be a closed form on the Lie algebra $\mathcal{G}$ with values in $\mathbb{R}^k$, and assume $\mathcal{S}_B \cap \mathcal{Z}(\mathcal{G}) = (0)$. Then the automorphism group $\mathrm{Aut}\,\mathcal{G}(B)$ of the extended algebra $\mathcal{G}(B)$ consists of all linear operators of the matrix form $\alpha = \left(\begin{array}{c|c} \alpha_o & 0 \\ \hline \varphi & \psi \end{array}\right)$ as in (2.2), where

$$(2.4) \qquad B(\alpha_o X, \alpha_o Y) = \psi B(X,Y) + \varphi[X,Y], \quad \text{all } X,Y \in \mathcal{G}.$$

2.5. Examples. The Heisenberg algebra $\mathcal{G}_3$ with non-zero brackets $[e_1,e_2] = e_3$ between the basis elements e_1,e_2,e_3 is a central extension of the abelian algebra $\mathcal{G} = \mathbb{R}e_1 \times \mathbb{R}e_2$ by $\mathbb{R}e_3$, given by the bilinear form $B = B_{12} : (\Sigma x_i e_i, \Sigma y_i e_i) \to x_1y_2 - x_2y_1$. Now $\mathcal{S}_B = (0)$ and $\mathrm{Aut}(\mathcal{G}_3)$ consists of all operators

$$(2.5) \qquad \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ u_1 & u_2 & c_o \end{pmatrix}, \text{ where } ad - bc = c_o \neq 0.$$

The four dimensional nilpotent Lie algebra $\mathcal{G}_4$ given by $[e_1,e_2] = e_3$, $[e_1,e_3] = e_4$ is a central extension of $\mathcal{G}_3$ by $\mathbb{R}e_4$, given by the bilinear form $B = B_{13} : (\sum_{i=1} x_i e_i, \sum_{i=1} y_i e_i) \to x_1y_3 - x_3y_1$. Thus $\mathcal{S}_B \cap \mathcal{Z}(\mathcal{G}_3) = (0)$, and $\mathrm{Aut}\,\mathcal{G}_4$ consists of all operators $\alpha = \left(\begin{array}{c|c} \alpha_o & 0 \\ \hline \varphi & c_1 \end{array}\right)$, $\alpha_o \in \mathrm{Aut}\,\mathcal{G}_3$, $\varphi \in \mathcal{G}_3^*$, $c_1 \in \mathbb{R}$, such that (2.4) is satisfied. This gives

$$(2.6) \qquad \alpha = \begin{pmatrix} a & 0 & 0 & 0 \\ c & d & 0 & 0 \\ u_1 & u_2 & ad & 0 \\ v_1 & v_2 & au_2 & a^2d \end{pmatrix}, \quad ad \neq 0.$$

On the basis of $\mathrm{Aut}\,\mathcal{G}_3$ and $\mathrm{Aut}\,\mathcal{G}_4$ one computes without difficulty the automorphism groups of all five-demensional nilpotent Lie algebras. These can be found in Table 4.3. (except for $\mathrm{Aut}(\mathcal{G}_{5,1})$ which is not needed).

3. Central extensions of Lie algebras.

3.1. We continue our study of central extensions $\mathcal{G}(B)$ of a Lie algebra $\mathcal{G}$ given by closed forms $B : \mathcal{G} \times \mathcal{G} \to \mathbb{R}^k$, and proceed to exclude forms B such that $\mathcal{G}(B) \cong \mathbb{R} \times \mathcal{G}(B')$, where $B' : \mathcal{G} \times \mathcal{G} \to \mathbb{R}^{k-1}$. Let J be the set of all linear maps $F : \mathbb{R}^k \to \mathbb{R}^k$ such that there exists a linear map $\varphi : \mathcal{G} \to \mathbb{R}^k$ with the property

(3.1) $\quad F(B(X,Y)) = \varphi[X,Y], \quad \text{all} \quad X,Y \in \mathcal{G}$.

Clearly J is a left ideal of $\mathrm{Hom}(\mathbb{R}^k,\mathbb{R}^k)$, i.e. $J = \mathrm{Hom}(\mathbb{R}^k,\mathbb{R}^k) \circ J$, and therefore J is generated by a projection π; $J = \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^k) \circ \pi$. We have

$$\pi(B(X,Y)) = \varphi_\pi[X,Y], \quad \varphi_\pi \in \mathrm{Hom}(\mathcal{G}, \mathbb{R}^k).$$

Put

(3.2) $\quad B'(X,Y) = B(X,Y) - \varphi_\pi[X,Y] = (1-\pi)B(X,Y)$,

in particular B' is cohomologous to B.

3.2. Lemma. Suppose we have an equation

$$F(B'(X,Y)) = \varphi[X,Y], \quad \text{all} \quad X,Y \in \mathcal{G},$$

where $\varphi \in \mathrm{Hom}(\mathcal{G}, \mathbb{R}^k)$ and B' is given by (3.2).
Then $\varphi[X,Y] = 0$, all $X,Y \in \mathcal{G}$.

Proof. If the above equality holds, then

$$F\circ(1-\pi)B(X,Y) = \varphi[X,Y], \quad \text{all} \quad X,Y \in \mathcal{G},$$

so that $F\circ(1-\pi) \in J$, and hence $F\circ(1-\pi) = G \circ \pi$, $G \in \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^k)$, and $F = (G+F) \circ \pi$; hence

$$F\circ(1-\pi) = (G+F)\circ\pi\circ(1-\pi) = (G+F)\circ(\pi-\pi) = 0 .$$

3.3. It follows from (3.2) that $\mathcal{G}(B)$ and $\mathcal{G}(B') = \mathcal{G}(B-\varphi_\pi[\cdot,\cdot])$ are isomorphic. Suppose $\pi \neq 0$ $(J \neq (0))$, then obviously

$$\mathcal{G}(B') = \mathcal{G}(B_o') \times \pi(\mathbb{R}^k)$$

where $\pi(\mathbb{R}^k)$ is abelian and

$$B_o' = (1-\pi)\circ B : \mathcal{G}\times\mathcal{G} \to (1-\pi)\,\mathbb{R}^k .$$

Corollary. Let $\mathcal{J}_B \cap \mathfrak{Z} = (0)$ and assume that $\mathcal{G}(B)$ can not be written as $\mathbb{R}\times\mathcal{G}$ for any Lie algebra $\mathcal{G}$. For any pair of linear maps $F: \mathbb{R}^k \to \mathbb{R}^k$, $\varphi: \mathcal{G} \to \mathbb{R}^k$ such that $F\circ B(X,Y) \equiv \varphi[X,Y]$, we have $F = 0$.

We can always assume the corollary holds, since we are not interested in algebras of the form $\mathbb{R}\times\mathcal{G}$.

3.4. In order to classify Lie algebras $\tilde{\mathcal{G}}$ with no factor isomorphic to $\mathbb{R}$, and with center $\tilde{\mathfrak{Z}}$ of dimension k and $\tilde{\mathcal{G}}/\tilde{\mathfrak{Z}} \cong \mathcal{G}$, we must consider closed forms $B: \mathcal{G}\times\mathcal{G} \to \mathbb{R}^k$ as in Corollary 3.3. Let S be the vector space of all closed forms on $\mathcal{G}$, and let S' be the subspace of all <u>exact</u> forms $B(X,Y) \equiv \varphi[X,Y]$ where $\varphi: \mathcal{G} \to \mathbb{R}$ is linear, i.e. $\varphi \in \mathcal{G}^*$. Then Corollary 3.3 is equivalent to the following: Let $\pi_1,\ldots,\pi_k : \mathbb{R}^k \to \mathbb{R}$ be the coordinate functionals, then $\pi_1\circ B,\ldots,\pi_k\circ B$ are linearly independent in S/S'. We know from § 2.3 that $\mathcal{G}(B_1) \cong \mathcal{G}(B_2) \iff B_2(\alpha_o X,\alpha_o Y) \equiv \psi B_1(X,Y) + \varphi[X,Y]$ where $\alpha = \left(\begin{array}{c|c} \alpha_o & 0 \\ \hline \varphi & \psi \end{array}\right)$ is an isomorphism. Such an identity holds if and only if $\pi_1\circ B_1,\ldots,\pi_k\circ B_1$ and $\pi_1\circ B_2\circ\alpha_o,\ldots,\pi_k\circ B_2\circ\alpha_o$ generate the same subspace of S/S'. We say that an $\mathrm{Aut}(\mathcal{G})$-orbit Ω in the set of all k-dimensional subspaces of S/S' <u>has no kernel in the center $\mathfrak{Z}$ of $\mathcal{G}$</u> if $\mathcal{J}_B \cap \mathfrak{Z} = (0)$ for some (and hence for all)

$B \in \Omega$. Up to this point we have made no use of the fact that $\mathfrak{g}$ was a real Lie algebra, thus we may state the following result for Lie algebras over arbitrary fields.

3.5. Theorem. Let $\mathfrak{g}$ be a Lie algebra over a field K. The isomorphism classes of Lie algebras $\tilde{\mathfrak{g}}$ with center $\tilde{\mathfrak{z}}$ of dimension k, $\tilde{\mathfrak{g}}/\tilde{\mathfrak{z}} \cong \mathfrak{g}$, and without abelian direct factors, are in bijective correspondence with those $\mathrm{Aut}(\mathfrak{g})$-orbits in the set of all k-dimensional subspaces of S/S' which have no kernel in $\mathfrak{z}$.

Let $\Lambda_2 \mathbb{R}^k$ be the set of all skew symmetric bilinear forms on $\mathbb{R}^k$ with values in $\mathbb{R}$, and let $G_n(V)$ = the set of all n-dimensional subspaces of a vector space V. As an application to the above theorem we get

3.6. Proposition. There are infinitely many mutually non-isomorphic nilpotent real Lie algebras of dimension 9.

Proof. We will consider only those algebras $\mathfrak{g}$ with $\dim \mathfrak{z} = 3$ and with $\mathfrak{g}/\mathfrak{z} \cong \mathbb{R}^6$. By Theorem 3.5 the isomorphism classes of such algebras are in bijective correspondence with certain orbits of $GL(6)$ in $G_3(\Lambda_2 \mathbb{R}^6)$. Letting U be the Zariski-open subset of $G_3(\Lambda_2 \mathbb{R}^6)$ of those 3-dimensional subspaces Q with $\mathfrak{z}_Q = (0)$, the orbits of $GL(6)$ in U are in bijective correspondence with the isomorphism classes of Lie algebras which we consider. We have

$$\dim U = \dim G_3(\Lambda_2 \mathbb{R}^6) = 3 \cdot (\tbinom{6}{2}-3) = 36$$

and $\dim GL(6) = 36$. However, since the center of $GL(6)$ is acting trivially, the orbits are of dimension ≤ 35. Since U is not a union of a finite number of analytic submanifolds of dimension less than $\dim U$, the proof is complete.

4. Six-dimensional nilpotent Lie algebras with center of dimension one.

4.1. By virtue of Theorem 3.5 we can, at least in principle, classify all nilpotent Lie algebras of dimension n, given all such algebras of dimension $< n$, and their automorphism groups. As an application we shall work out this program for $n = 6$, and we start with the class of algebras having one-dimensional center. According to [1] the algebras of dimension 5 which we have to extend are the following

$$\mathcal{G}_3 \times \mathbb{R}^2 , \quad \mathcal{G}_4 \times \mathbb{R}, \quad \mathcal{G}_{5,1}, \ \mathcal{G}_{5,2}, \ \mathcal{G}_{5,3}, \ \mathcal{G}_{5,4}, \ \mathcal{G}_{5,5}, \ \mathcal{G}_{5,6} .$$

We shall illustrate the computations by means of two examples. The remaining six algebras are treated analogously, and the result, **thirteen** new algebras, is listed in the table 4.3.

If $\{e_1,\ldots,e_n\}$ is a fixed basis for the Lie algebra $\mathcal{G}$, we let $B_{ij} : (\sum_{i=1}^{n} x_i e_i, \sum_{i=1}^{n} y_i e_i) \to x_i y_j - x_j y_i$, $1 \leq i < j \leq n$, denote the elementary bilinear forms.

4.2. Extensions of $\mathcal{G}_{5,4}$. $\mathcal{G}_{5,4}$ is given by the following non-zero bracket-relations between the elements of a basis:

$$[e_1,e_2] = e_3 , \quad [e_1,e_3] = e_4 , \quad [e_2,e_3] = e_5 .$$

We first compute $\mathrm{Aut}(\mathcal{G}_{5,4})$. Now $\mathcal{G}_{5,4}$ is a central extension of the Heisenberg algebra $\mathcal{G}_3$ by $\mathbb{R}e_4 \times \mathbb{R}e_5$ determined by the bilinear form $B = (B_{13},B_{23})$, and hence $\mathfrak{I}_B \cap \mathfrak{Z}(\mathcal{G}_3) = (0)$. By (2.2) and (2.5) every $\alpha \in \mathrm{Aut}\,\mathcal{G}_{5,4}$ can be written

$$\alpha = \left(\begin{array}{c|cc} \alpha_0 & 0 & \\ \hline \varphi & b_0 & \beta_0 \\ & c_0 & \gamma_0 \end{array}\right) ; \quad \alpha_0 = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ u_1 & u_2 & e \end{pmatrix} \in \mathrm{Aut}(\mathcal{G}_3), \ e = ad-bc \neq 0 , \quad \varphi \in \mathrm{Hom}(\mathcal{G}_3,\mathbb{R}^2) .$$

Computing $\alpha[X,Y]$ and $[\alpha X,\alpha Y]$ (or using (2.4)) we find that

$$\alpha = \begin{pmatrix} a & b & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 \\ u_1 & u_2 & e & 0 & 0 \\ b_1 & b_2 & au_2-bu_1 & ae & be \\ c_1 & c_2 & cu_2-du_1 & ce & de \end{pmatrix}, \quad ad-bc = e \neq 0 .$$

Next we find a basis for the set S of all closed bilinear forms on $\mathcal{G}_{5,4}$, and we can compute modulo the exact forms S'. Thus B_{12}, B_{13}, B_{23} is a basis for the space S' checking the Jacobi-identity (2.1) on all the remaining 7 elementary forms B_{ij} we find the following basis for S/S' : B_{25}, B_{14}, $B_{24}+B_{15}$.
Before computing orbits under $\mathrm{Aut}(\mathcal{G}_{5,4})$ we recall that only the forms B with $\mathcal{J}_B \cap \mathcal{Z}(\mathcal{G}_{54}) = (0)$ give new algebras with one-dimensional centers, in particular B_{25} and B_{14} do not satisfy this. For $\alpha \in \mathrm{Aut}(\mathcal{G}_{5,4})$ we see (regarding the forms as matrices)

$$\alpha^t(B_{24}+B_{15})\alpha = 2\,e\,a\,c\,B_{14} + 2\,e\,b\,d\,B_{25} + e(ad+bc)(B_{24}+B_{15}), \text{(modulo } S'),$$

Hence $B_{14}+B_{25}$ is not in the orbit of $B_{24}+B_{15}$, however $B_{14}+B_{25}$ satisfies $\mathcal{J}_B \cap \mathcal{Z}(\mathcal{G}_{54}) = (0)$. Now

$$\alpha^t(B_{14}+B_{25})\alpha = (a^2+c^2)\,e\,B_{14} + (b^2+d^2)\,e\,B_{25} + (ab+cd)e(B_{24}+B_{15}), \quad \text{(modulo } S'),$$

and

$$\alpha^t B_{25}\,\alpha = e\,c^2 B_{14} + e\,d^2 B_{25} + c\,d\,e(B_{24}+B_{15}), \quad \text{(modulo } S').$$

In particular the orbits $\Omega(B_{25})$ and $\Omega(B_{14})$ are identical and this orbit forms a 2-dimensional surface separating the two open orbits $\Omega(B_{15}+B_{24})$ and $\Omega(B_{14}+B_{25})$. The two last orbits correspond to two different isomorphism classes of six-dimensional algebras, and these are the only classes arising from $\mathcal{G}_{5,4}$. Thus, within isomorphisms, the only extensions of $\mathcal{G}_{5,4}$ are given by the bracket relations of

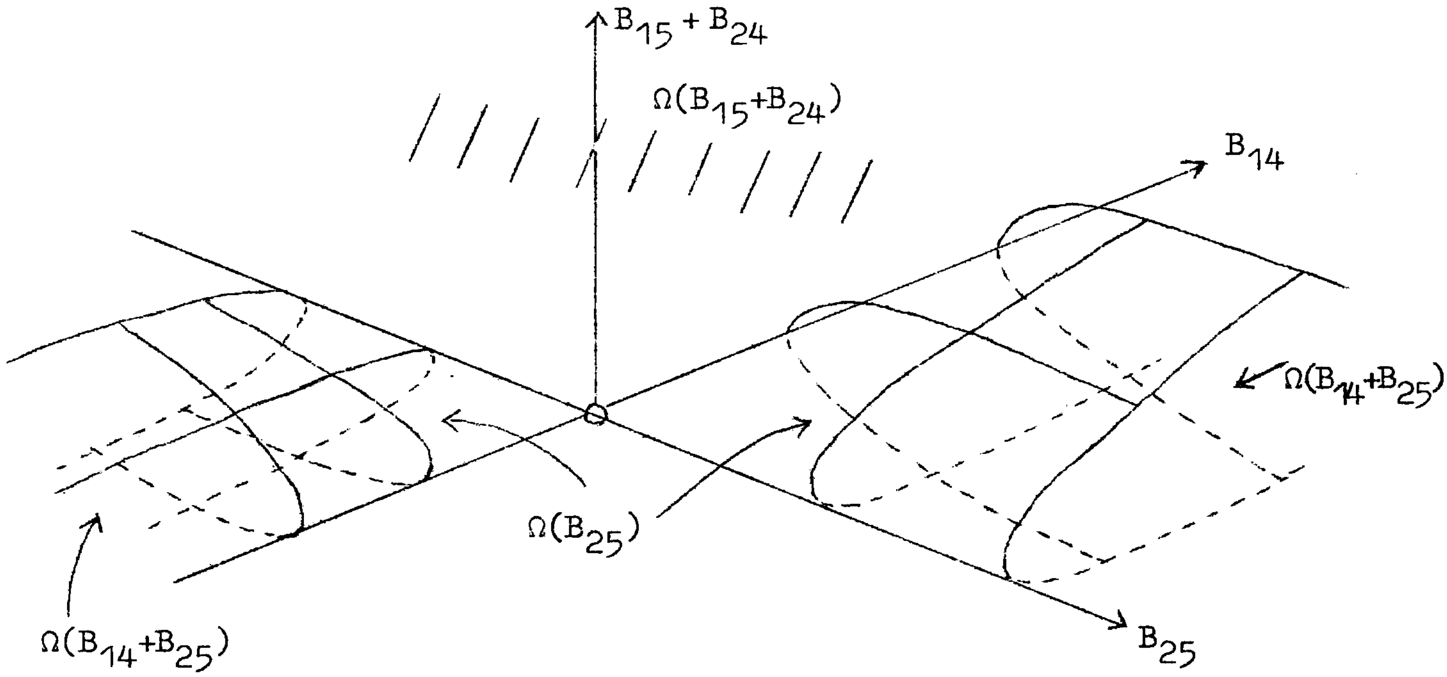
$\mathcal{G}_{5,4}$ together with the additional brackets,

$$[e_1,e_5] = [e_2,e_4] = e_6 \qquad (\Omega(B_{15}+B_{24}))$$

and

$$[e_1,e_4] = [e_2,e_5] = e_6 \qquad (\Omega(B_{14}+B_{25})).$$

We sketch the orbit space in S/S' :



(4.1) Orbits in $S/S' \cong H^2(\mathcal{G}_{5,4}, \mathbb{R})$ under $\mathrm{Aut}(\mathcal{G}_{5,4})$.

<u>4.2. Extensions of $\mathcal{G}_{5,6}$.</u> $\mathcal{G}_{5,6}$ is given by the non-zero brackets

$$[e_1,e_2] = e_3,\ [e_1,e_3] = e_4,\ [e_1,e_4] = e_5,\ [e_2,e_3] = e_5.$$

Hence $\mathcal{G}_{56}$ is a central extension of $\mathcal{G}_4$ by $\mathbb{R}e_5$, and every $\alpha \in \mathrm{Aut}(\mathcal{G}_{5,6})$ is of the form (2.2) where $\alpha_o \in \mathrm{Aut}(\mathcal{G}_4)$ is determined by (2.6).

From the relation (2.4) we derive

$$\alpha = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ c & a^2 & 0 & 0 & 0 \\ u_1 & u_2 & a^3 & 0 & 0 \\ b_1 & b_2 & au_2 & a^4 & 0 \\ c_1 & c_2 & ab_2+cu_2-a^2u_1 & a^2u_2+ca^3 & a^5 \end{pmatrix}, \; a \neq 0.$$

A basis for S/S' is $\{B_{34}+B_{25}, B_{24}+B_{15}\}$, and a calculation of the action of $\mathrm{Aut}(\mathcal{G}_{56})$ on each of these forms yields two orbits

$$\Omega = \{s(B_{15}+B_{24}) + t(B_{34}+B_{25}) : t \neq 0\}$$

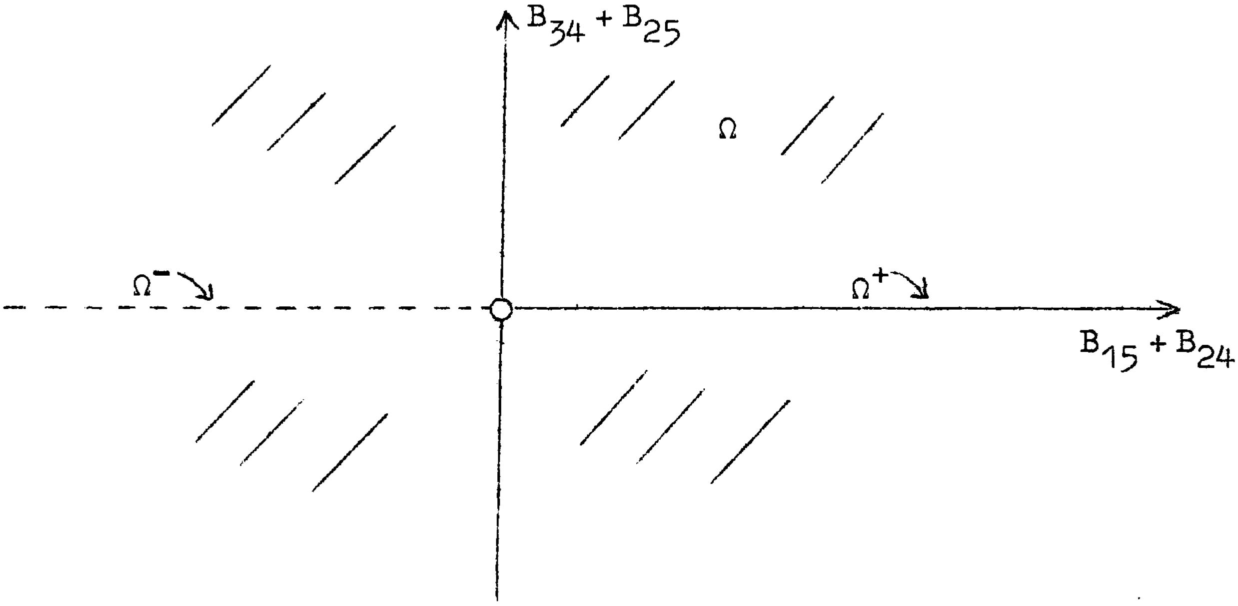
$$\Omega^+ = \{s(B_{15}+B_{24}) : s > 0\} .$$

In addition, we have the orbit $\Omega^- = \{s(B_{15}+B_{24}) : s < 0\}$. Of course, Ω^+ and Ω^- define the same isomorphism class of algebras. Thus, within isomorphisms, there are two extensions of $\mathcal{G}_{5,6}$ with one-dimensional center. They are given by the bracket relations of $\mathcal{G}_{5,6}$ and the additional relations:

$$[e_3,e_4] = [e_2,e_5] = e_6 \qquad (\Omega)$$

and

$$[e_1,e_5] = [e_2,e_4] = e_6 \qquad (\Omega^+)$$



(4.2) Orbits in $S/S' \cong H^2(\mathcal{G}_{5,6}, \mathbb{R})$ under $\mathrm{Aut}(\mathcal{G}_{5,6})$.

<u>4.3. Table</u>. Six-dimensional real nilpotent Lie algebras $\tilde{\mathcal{G}}$ with one-dimensional center $\tilde{\mathfrak{z}}$. $\tilde{\mathcal{G}}/\tilde{\mathfrak{z}} = \mathcal{G}$.

$\mathcal{G}$	Basis for the set of closed non-exact forms on $\mathcal{G}$ $(H^2(\mathcal{G}))$	Aut($\mathcal{G}$)	Representative for defining orbit in $H^2(\mathcal{G})$	Product in $\tilde{\mathcal{G}}$. Only non-zero brackets are given.	$\tilde{\mathcal{G}}$
$\mathcal{G}_3\times\mathbb{R}^2$	$B_{13},B_{23},B_{24},$ $B_{25},B_{14},B_{15},$ B_{45}	$\begin{pmatrix} a & b & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 \\ b_1 & b_2 & c_o & b_3 & b_4 \\ c_1 & c_2 & 0 & c_3 & c_4 \\ u_1 & u_2 & 0 & u_3 & u_4 \\ \multicolumn{5}{c}{c_o = ad-bc \neq 0} \\ \multicolumn{5}{c}{c_3u_4-u_3c_4 \neq 0} \end{pmatrix}$	$B_{13}+B_{45}$	$[e_1,e_2]=e_3$ $[e_1,e_3]=$ $[e_4,e_5]=e_6$	$\mathcal{G}_{6,1}$
$\mathcal{G}_4\times\mathbb{R}$	$B_{15},B_{25},$ B_{14},B_{23}	$\begin{pmatrix} a & 0 & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 \\ u_1 & u_2 & ad & 0 & 0 \\ b_1 & b_2 & au_2 & a^2d & a_o \\ v_1 & v_2 & 0 & 0 & c_o \end{pmatrix}$ $adc_o \neq 0$	$B_{14}+B_{25}$	$[e_1,e_2]=e_3$ $[e_1,e_3]=e_4$ $[e_1,e_4]=$ $[e_2,e_5]=e_6$	$\mathcal{G}_{6,2}$
			$B_{14}+B_{25}+B_{23}$	$[e_1,e_2]=e_3$ $[e_1,e_3]=e_4$ $[e_1,e_4]=$ $[e_2,e_5]=$ $[e_2,e_3]=e_6$	$\mathcal{G}_{6,3}$
$\mathcal{G}_{5,1}$	B_{13},B_{14} B_{23},B_{24}		-	-	-
$\mathcal{G}_{5,2}$	$B_{14},B_{15},B_{24},$ $B_{35},B_{25}+B_{34},$ B_{23}	$\begin{pmatrix} a & 0 & 0 & 0 & 0 \\ d & e & f & 0 & 0 \\ g & h & k & 0 & 0 \\ u_1 & u_2 & u_3 & ae & af \\ v_1 & v_2 & v_3 & ah & ak \end{pmatrix}$ $a \neq 0,\ ek \neq hf$	$B_{25}+B_{34}$	$[e_1,e_2]=e_4$ $[e_1,e_3]=e_5$ $[e_2,e_5]=$ $[e_3,e_4]=e_6$	$\mathcal{G}_{6,4}$
			$B_{35}+B_{14}$	$[e_1,e_2]=e_4$ $[e_1,e_3]=e_5$ $[e_1,e_4]=$ $[e_3,e_5]=e_6$	$\mathcal{G}_{6,5}$
			$B_{24}+B_{35}$	$[e_1,2_2]=e_4$ $[e_1,e_3]=e_5$ $[e_2,e_4]=$ $[e_3,e_5]=e_6$	$\mathcal{G}_{6,6}$

$\mathcal{G}_{5,3}$	$B_{15}-B_{34}$ B_{24}, B_{13}	$\begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 & 0 \\ u_1 & u_2 & a^2 & 0 & 0 \\ v_1 & v_2 & 0 & ad & 0 \\ b_1 & b_2 & b_3 & b_4 & a^2d \end{pmatrix}$ $b_4 = av_2-du_1,\ ad \neq 0$	$B_{15}-B_{34}$	$[e_1,e_2] = e_4$ $[e_1,e_4] =$ $[e_2,e_3] = e_5$ $[e_1,e_5] =$ $-[e_3,e_4] = e_6$	$\mathcal{G}_{6,7}$
$\mathcal{G}_{5,4}$	B_{14}, B_{25}, $B_{15}+B_{24}$	$\begin{pmatrix} a & b & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 \\ u_1 & u_2 & e & 0 & 0 \\ b_1 & b_2 & b_3 & ae & be \\ c_1 & c_2 & c_3 & ce & de \end{pmatrix}$	$B_{15}+B_{24}$	$[e_1,e_2] = e_3$ $[e_1,e_3] = e_4$ $[e_2,e_3] = e_5$ $[e_1,e_5] =$ $[e_2,e_4] = e_6$	$\mathcal{G}_{6,8}$
		$e = ad-bc \neq 0$ $b_3 = au_2-bu_1$, $c_3 = cu_2-du_1$	$B_{25}+B_{14}$	$[e_1,e_2] = e_3$ $[e_1,e_3] = e_4$ $[e_2,e_3] = e_5$ $[e_2,e_5] =$ $[e_1,e_4] = e_6$	$\mathcal{G}_{6,9}$
$\mathcal{G}_{5,5}$	B_{15}, B_{23}	$\begin{pmatrix} a & 0 & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 \\ u_1 & u_2 & ad & 0 & 0 \\ b_1 & b_2 & au_2 & a^2d & 0 \\ c_1 & c_2 & ab_2 & a^2u_2 & a^3d \end{pmatrix}$	B_{15}	$[e_1,e_2] = e_3$ $[e_1,e_3] = e_4$ $[e_1,e_4] = e_5$ $[e_1,e_5] = e_6$	$\mathcal{G}_{6,10}$
		$ad \neq 0$	$B_{15}+B_{23}$	$[e_1,e_2] = e_3$ $[e_1,e_3] = e_4$ $[e_1,e_4] = e_5$ $[e_1,e_5] =$ $[e_2,e_3] = e_6$	$\mathcal{G}_{6,11}$
$\mathcal{G}_{5,6}$	$B_{24}+B_{15}$ $B_{25}+B_{34}$	$\begin{pmatrix} a & 0 & 0 & 0 & 0 \\ c & a^2 & 0 & 0 & 0 \\ u_1 & u_2 & a^3 & 0 & 0 \\ b_1 & b_2 & au_2 & a^4 & 0 \\ c_1 & c_2 & c_3 & c_4 & a^5 \end{pmatrix}$	$B_{24}+B_{15}$	$[e_1,e_2] = e_3$ $[e_1,e_3] = e_4$ $[e_1,e_4] =$ $[e_2,e_3] = e_5$ $[e_2,e_4] =$ $[e_1,e_5] = e_6$	$\mathcal{G}_{6,12}$
			$B_{34}+B_{25}$	$[e_1,e_2] = e_3$ $[e_1,e_3] = e_4$ $[e_1,e_4] =$ $[e_2,e_3] = e_5$ $[e_3,e_4] =$ $[e_2,e_5] = e_6$	$\mathcal{G}_{6,13}$

5. Six-dimensional nilpotent Lie algebras with center of dimension ≥ 2.

5.1. Let $G_k(V)$ be the set of k-dimensional linear subspaces of a linear space V. Then by Theorem 3.5, the orbits of $\mathrm{Aut}(\mathcal{G})$ in $G_k(H^2(\mathcal{G}))$ determine isomorphism classes of Lie algebras $\tilde{\mathcal{G}}$ with $\dim \tilde{\mathcal{G}} = \dim \mathcal{G} + k$. Here $G_k(V)$ is a real-analytic compact manifold, and the action of $\mathrm{Aut}(\mathcal{G})$ is real-analytic. We will consider the case

$$\dim \tilde{\mathcal{G}} = 6, \quad \dim \mathcal{G} \leq 4,$$

assuming that $\tilde{\mathcal{G}}$ does not have a direct abelian factor algebra. If $\dim \mathcal{G} \leq 3$, then $S(\mathcal{G}) \subset \Lambda_2(\mathcal{G})$ implies that $\dim H^2(\mathcal{G}) \leq \dim \Lambda_2(\mathcal{G}) \leq 3$ with equality only for $\mathcal{G} = \mathbb{R}^3$. Thus we obtain the algebra $\tilde{\mathcal{G}} = \mathbb{R}^3(B_{12}, B_{23}, B_{13})$. When $\dim \mathcal{G} = 4$, we have $\mathcal{G} = \mathcal{G}_4$, $\mathbb{R} \times \mathcal{G}_3$, or $\mathbb{R}^4$.

5.2. $\mathcal{G} = \mathcal{G}_4$: The product in $\mathcal{G}_4$ is $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, and hence $S'(\mathcal{G}_4) = (B_{12}, B_{13}) \subset S(\mathcal{G}_4) = (B_{12}, B_{13}, B_{14}, B_{23})$, which gives the unique algebra $\tilde{\mathcal{G}} = \mathcal{G}_4(B_{14}, B_{23})$ of dimension 6.

5.3. $\mathcal{G} = \mathbb{R}^4$. In this case we have $H^2(\mathbb{R}^4) = \Lambda_2 \mathbb{R}^4$ of dimension 6 with basis B_{ij}, $1 \leq i < j \leq 4$, and

$$\mathrm{Aut}(\mathbb{R}^4) = GL(4).$$

$G_2(\Lambda_2 \mathbb{R}^4)$ is the union of 4 disjoint $GL(4)$-invariant sets Ω_i, $1 \leq i \leq 4$. We first define the Ω_i, and then show that they actually are orbits of $GL(4)$. Let P be a two-dimensional subspace of $\Lambda_2 \mathbb{R}^4$.

$$\Omega_1 = \{P : \mathfrak{J}_P \neq (0)\}$$

$$\Omega_2 = \{P : \mathfrak{J}_P = (0) \text{ and } P \text{ contains forms } B, B' \text{ with } B \text{ of rank } 2 \text{ and } B' | \mathfrak{J}_B \neq 0\}$$

$\Omega_3 = \{P : \mathcal{S}_P = (0)$ and $P = (B,B')$ where B is of rank 2 and $B' | \mathcal{S}_B = 0\}$

$\Omega_4 = \{P :$ any $B \in P$ is nondegenerate or trivial$\}$.

<u>5.3.1.</u> Clearly $G_2(\Lambda_2 \mathbb{R}^4)$ is the union of the Ω_i. To show that Ω_1 is an orbit, let $P \in \Omega_1$. Then $\dim \mathcal{S}_P = 1$, and hence, setting $\mathbb{R}^3 = \mathbb{R}^4/\mathcal{S}_P$, we obtain a plane $P' \in G_2(\Lambda_2 \mathbb{R}^3)$. The exterior product $(\Lambda_2 \mathbb{R}^3) \otimes (\mathbb{R}^3)^* = \Lambda^2(\mathbb{R}^3)^* \otimes (\mathbb{R}^3)^* \to \Lambda^3(\mathbb{R}^3)^* = \mathbb{R}$, defines a $GL(3)$-invariant isomorphism $G_2(\Lambda_2 \mathbb{R}^3) = G_2(\mathbb{R}^3)$, and hence $GL(3)$ is acting transitively on $G_2(\Lambda_2 \mathbb{R}^3)$. Also $GL(4)$ is acting transitively on the set of lines through the origin in $\mathbb{R}^4$, such as $\mathcal{S}_P$, and hence it follows that $GL(4)$ is acting transitively on Ω_1. By the same reasoning, $SO(4)$ is acting transitively on Ω_1, and one can show that $\Omega_1 = SO(4)/\mathbb{Z}_2 \times O(2)$.

<u>5.3.2.</u> For $P \in \Omega_2$, we will choose a basis for $\mathbb{R}^4$ such that there is a basis B, B' of P which has a standard matrix form in terms of the basis for $\mathbb{R}^4$. It will follow that Ω_2 is an orbit of $GL(4)$. To find such a basis, let B be of rank 2, and $B' | \mathcal{S}_B \neq 0$. Choose $e_i \in \mathbb{R}^4$ such that $\mathcal{S}_B = (e_3, e_4)$ and $\operatorname{Ker} B'(e_3,-) \cap \operatorname{Ker} B'(e_4,-) = (e_1, e_2)$. Since $B' | \mathcal{S}_B$ is nonsingular, we have a basis for $\mathbb{R}^4$, in terms of which $B = a B_{12}$ and $B' = b B_{12} + c B_{34}$, $ac \neq 0$. Hence P has standard form $P = (B_{12}, B_{34})$. It follows that the isotropy group of P has identity component $(GL(2) \times GL(2))^0$, and hence that $\dim \Omega_2 = 16 - 8 = 8 = \dim G_2(\Lambda_2 \mathbb{R}^4)$ so that Ω_2 is an open orbit. Because (B_{12}, B_{34}) is invariant under $A = \operatorname{diag}(-1,1,1,1)$, it follows that Ω_2 is connected.

<u>5.3.3.</u> When $P \in \Omega_3$, we have $P = (B,B')$ where B is of rank 2 and $B' | \mathcal{S}_B = 0$. We will find a suitable basis for $\mathbb{R}^4$.

If $\mathcal{J}_{B'} + \mathcal{J}_B = \mathbb{R}^4$, we clearly have $B' = 0$. Hence, if $\dim \mathcal{J}_{B'} = 2$, we obtain $\mathcal{J}_P = \mathcal{J}_{B'} \cap \mathcal{J}_B \neq (0)$, contrary to the definition of Ω_3. It follows that $\dim \mathcal{J}_{B'} = 0$, that is, B' is nondegenerate, and we can choose a basis for $\mathbb{R}^4$ such that $\mathcal{J}_B = (e_3, e_4)$ and $\operatorname{Ker} B'(e_3,-) = (e_3, e_4, e_2)$, $B'(e_1, e_3) = 1$, $\operatorname{Ker} B'(e_4,-) = (e_3, e_4, e_1)$, and $B'(e_2, e_4) = 1$.

Then $B = aB_{12}$, $B' = B_{13} + B_{24}$ and $P = (B_{12}, B_{13}+B_{24})$ and hence Ω_3 is a single orbit. One can show that the Lie algebra of the isotropy group of P (the isotropy algebra, see 5.4.3) is

$$\mathcal{G}_P = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline B & cI - A^t \end{array} \right) : A, B \in gl(2),\ c \in \mathbb{R} \right\},$$

and hence that $\dim \Omega_3 = 16 - 9 = 7$. Ω_3 has two components because the orientation of $\mathbb{R}^4$ defined by the chosen basis (e_1, e_2, e_3, e_4) depends only on P.

<u>5.4.3.</u> In order to show that Ω_4 is a single orbit, we will do as follows: First we will find a connected subset C of Ω_4 such that every $GL(4)$-orbit in Ω_4 meets C. Next we will show that each point in C lies in an open $GL(4)$-orbit. It then follows that C lies in a single orbit, which now must equal Ω_4. To find the set C, let $P = (B, B') \in \Omega_4$, and choose a plane $\pi_1 \subset \mathbb{R}^4$ such that $B|\pi_1 \neq 0$ and $B'|\pi_1 \neq 0$. Then there are unique planes $\pi_2, \pi_3 \subset \mathbb{R}^4$ such that $B(\pi_1, \pi_2) = 0$ and $B'(\pi_1, \pi_3) = (0)$. There is a number c such that $B'|\pi_1 = cB|\pi_1$. It follows that $(B' - cB)|(\pi_1 + (\pi_2 \cap \pi_3)) = 0$. Since $B' - cB$ is nondegenerate, $\dim(\pi_1 + (\pi_2 \cap \pi_3)) \leq 2$, and hence $\pi_2 \cap \pi_3 = (0)$. It follows that $\pi_i \cap \pi_j = (0)$ for $1 \leq i < j \leq 3$.

<u>Lemma.</u> $GL(4)$ is acting transitively on the set of triples (π_1, π_2, π_3) where π_i are planes in $\mathbb{R}^4$ such that $\pi_i \cap \pi_j = (0)$ for $i \neq j$.

Proof. Clearly, $GL(4)$ is acting transitively on the set of pairs (π_1,π_2) where $\pi_1 \cap \pi_2 = (0)$. Hence it suffices to show that $GL(2)^2 = GL(\pi_1) \times GL(\pi_2)$ is acting transitively on the set of all planes π_3 with $\pi_3 \cap \pi_1 = \pi_3 \cap \pi_2 = (0)$. Because $\pi_3 \cap \pi_2 = (0)$, there is a linear map $\varphi : \pi_1 \to \pi_2$ such that

$$\pi_3 = \{v + \varphi(v) : v \in \pi_1\} ,$$

and φ is an isomorphism because $\operatorname{Ker}\varphi = \pi_1 \cap \pi_3 = (0)$.
For $(A_1,A_2) \in GL(2)^2 \subset GL(4)$, we have

$$(A_1,A_2)(\pi_3) = \{A_1 v + A_2\varphi(v) : v \in \pi_1\} = \{v + A_2\varphi A_1^{-1}(v) : v \in \pi_1\} ,$$

and since the action $(A_1,A_2)\varphi = A_2\varphi A_1^{-1}$ on the set of linear isomorphisms $\varphi : \pi_1 \to \pi_2$ is transitive, the proof is complete.

It follows from the lemma that there is some $P' = (B_1,B_2)$ in the $GL(4)$-orbit of P such that $B_1(\pi_1,\pi_2) = (0)$ and $B_2(\pi_1,\pi_3) = (0)$ where

$$\pi_1 = (e_3,e_4), \quad \pi_2 = (e_1,e_2), \quad \text{and} \quad \pi_3 = (e_1+e_3,\ e_2+e_4) .$$

The only B_1,B_2 satisfying this condition are

$$B_1 = a B_{12} + b B_{34}, \quad ab \neq 0, \quad \text{and}$$
$$B_2 = c B_{12} + d(B_{12}+B_{23}-B_{14}), \quad d \neq 0,$$

where B_2 is nondegenerate for $d \neq 0$. It follows that

$$P' = (B_{12}+b'B_{34}\ ,c'B_{34}+B_{23}-B_{14}), \quad b' \neq 0 .$$

Choose $x > 0$ and $\varepsilon = \pm 1$ such that $x^2 b' = -\varepsilon$ and set $\alpha = -xc'$. Then the matrix $\operatorname{diag}(1,1,x,-x)$ will transform P' to

$$P'' = (B_{12}+\varepsilon B_{34},\alpha B_{34}+B_{23}+B_{14}) = P(\alpha,\varepsilon) .$$

$$\det(s(B_{12}+\varepsilon B_{34}) + t(\alpha B_{34}+B_{23}+B_{14})) = (s(\varepsilon s+\alpha t) + t^2)^2 =$$
$$= ((t+\alpha s/2)^2 - (\alpha^2-4\varepsilon)s^2/4)^2 .$$

This polynomial in s and t contains a linear factor if and only if $\alpha^2 \geq 4\epsilon$, and $P(\alpha,\epsilon)$ will then contain a degenerate form. Hence there is a connected subset C of Ω_4,

$$C = \{P(\alpha,1) : |\alpha| < 2\} = \{P(\alpha,\epsilon) : P(\alpha,\epsilon) \in \Omega_4\}$$

To complete the proof that Ω_4 is a single orbit, it suffices to show that each of the planes $P(\alpha,1)$, $|\alpha| < 2$, lies on an open orbit in $G_2(\Lambda_2 \mathbb{R}^4)$.
For $B \in \Lambda_2 \mathbb{R}^4$ and $A \in gl(4)$, we set $B^A(v,w) = B(e^{tA}v, e^{tA}w)'_0 = B(Av,w) + B(v,Aw)$. Then the isotropy algebra of a plane $P \subset \Lambda_2 \mathbb{R}^4$ is

$$\mathcal{G}_P = \{A \in gl(4) : B^A \in P \text{ for all } B \in P\}.$$

Setting $P = (B_1, B_2)$, the conditions $B_1^A, B_2^A \in P$ are a system of linear equations in the entries of A, and the rank of this system equals $\dim gl(4) - \dim \mathcal{G}_P$, which is equal to $\dim \Omega(P)$. To write down those equations for the plane $P(\alpha,\epsilon)$, we note that, with $A = (a_{ij})$,

$$(B_{ij})^A = \sum_{k=1}^{4} (a_{ik} B_{kj} - a_{jk} B_{ki}).$$

Setting $B_1 = B_{12} + \epsilon B_{34}$ and $B_2 = \alpha B_{34} + B_{23} + B_{14}$, we have

$$\begin{aligned} B_1^A = {} & (a_{23} - \epsilon a_{41}) B_{13} + (-a_{14} + \epsilon a_{32}) B_{24} \\ & + (a_{11} + a_{22}) B_{12} + \epsilon (a_{33} + a_{44}) B_{34} \\ & - (a_{23} + \epsilon a_{42}) B_{23} + (a_{24} + \epsilon a_{31}) B_{14}, \end{aligned}$$

and

$$\begin{aligned} B_2^A = {} & (-\alpha a_{41} + a_{21} + a_{43}) B_{13} + (\alpha a_{32} + a_{34} + a_{12}) B_{24} \\ & + (-a_{31} + a_{42}) B_{12} + (\alpha a_{33} + \alpha a_{44} - a_{24} + a_{13}) B_{34} \\ & + (-\alpha a_{42} + a_{22} + a_{33}) B_{23} + (\alpha a_{31} + a_{11} + a_{44}) B_{14}. \end{aligned}$$

The condition $B_1^A \in (B_1,B_2)$ is expressed by the 4 equations

(1) $a_{23} - \varepsilon a_{41} = 0$ (2) $a_{32} - \varepsilon a_{14} = 0$

(3) $a_{24} + \varepsilon a_{31} + a_{13} + \varepsilon a_{42} = 0$

(4) $a_{33} + a_{44} - a_{11} - a_{22} + \alpha\varepsilon\, a_{13} + \alpha a_{42} = 0$,

and the condition $B_2^A \in (B_1,B_2)$ by

(5) $a_{21} - \alpha a_{41} + a_{43} = 0$ (6) $a_{12} + \alpha a_{32} + a_{34} = 0$

(7) $a_{11} - a_{22} + a_{44} - a_{33} + \alpha a_{42} + \alpha a_{31} = 0$

(8) $a_{24} - \varepsilon a_{31} + \alpha a_{22} - \alpha a_{44} - a_{13} + (\varepsilon - \alpha^2) a_{42} = 0$

The system S_1 formed by the equations numbered (1), (2), (5), and (6) involve only the 8 variables a_{12}, a_{21}, a_{23}, a_{32}, a_{14}, a_{41}, a_{34}, and a_{43} and this system is of rank 4.
The system S_2 formed by the equations numbered (3), (4), (7), and (8) involve only the 8 remaining variables a_{13}, a_{31}, a_{24}, a_{42}, and a_{ii}. The linear space generated by the coefficient couloumns of S_2 is generated by the coefficient couloumns of a_{24}, a_{11}, a_{22}, and a_{13}, and the coefficient matrix of those variables is

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & \varepsilon\alpha \\ 0 & 1 & -1 & 0 \\ 1 & 0 & \alpha & -1 \end{pmatrix} \quad \text{with } \det M = \varepsilon\alpha^2 - 4 .$$

Hence S_2 has rank 4 if $\alpha^2 \neq 4\varepsilon$, and has rank 3 if $\alpha^2 = 4\varepsilon$, and the whole system (1) through (8) has rank 8 if $\alpha^2 \neq 4\varepsilon$ and has rank 7 if $\alpha^2 = 4\varepsilon$.
It follows that $\dim \Omega(P(\alpha,\varepsilon)) = 8$ when $\alpha^2 \neq 4\varepsilon$, and hence that each $P(\alpha,\varepsilon)$ contained in C lies on an open orbit, completing the proof.

<u>5.4. Theorem.</u> $G_2(\Lambda_2\mathbb{R}^4)$ is the union of four $GL(4)$-orbits Ω_i, $1 \le i \le 4$, such that

$$(B_{12},B_{23}) \in \Omega_1\,, \quad \dim\Omega_1 = 5\,,$$
$$(B_{12},B_{34}) \in \Omega_2\,, \quad \dim\Omega_2 = 8\,,$$
$$(B_{12},B_{13}+B_{24}) \in \Omega_3\,, \quad \dim\Omega_3 = 7\,, \text{ and}$$
$$(B_{12}+B_{34}, B_{23}+B_{14}) \in \Omega_4\,, \quad \dim\Omega_4 = 8\,.$$

<u>Remark.</u> To determine the orbit Ω_i to which a given plane $P \subset \Lambda_2\mathbb{R}^4$ belongs, one can proceed as follows. Let e_i, $1 \le i \le 4$ be the standard basis of $\mathbb{R}^4$ and define a symmetric bilinear form on $\Lambda_2\mathbb{R}^4$ by $\langle B_1,B_2\rangle = \frac{1}{4}\sum_\sigma \mathrm{sgn}(\sigma)B_1(e_{\sigma(1)},e_{\sigma(2)})B_2(e_{\sigma(3)},e_{\sigma(4)})$, the summation extending over all permutations of $\{1,2,3,4\}$. In terms of the basis B_{ij} for $\Lambda_2\mathbb{R}^4$, we have $\langle B_{\sigma(1)\sigma(2)},B_{\sigma(3)\sigma(4)}\rangle = \mathrm{sgn}(\sigma)$ and $\langle B_{ij},B_{jk}\rangle = 0$, and hence $\langle\cdot,\cdot\rangle$ is a nondegenerate symmetric form on $\Lambda_2\mathbb{R}^4$ of signature $(3,3)$. Restricting this form to P, we obtain $\langle\cdot,\cdot\rangle_P$, and the orbit containing P is characterized as follows.

$$\langle\cdot,\cdot\rangle_P = 0 \iff P \in \Omega_1\,,$$
$$\langle\cdot,\cdot\rangle_P \text{ is of rank one} \iff P \in \Omega_3\,,$$
$$\langle\cdot,\cdot\rangle_P \text{ is nonsingular, indefinite} \iff P \in \Omega_2\,,$$
$$\langle\cdot,\cdot\rangle_P \text{ is nonsingular and definite} \iff P \in \Omega_4\,.$$

The orbits Ω_3 and Ω_4 both have two components corresponding to those P with $\langle\cdot,\cdot\rangle_P$ positive or negative semidefinite.

<u>5.5. $\mathcal{G} = \mathbb{R}\times\mathcal{G}_3$:</u> $[e_1,e_2] = e_3$, e_4 is central,

$$S(\mathcal{G}) = (B_{13},B_{23},B_{14},B_{24},B_{12})\,, \quad S'(\mathcal{G}) = (B_{12})\,.$$

The automorphisms of $\mathcal{G}$ have matrix form

$$A = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ r & s & \delta & \beta \\ t & u & 0 & \alpha \end{pmatrix}, \quad \delta = ad - bc .$$

The automorphism induced by A in $H^2(\mathcal{G})$ does not depend on $r, s, t,$ and u. Hence, it suffices to take $A = \varphi\psi = \psi\varphi$ where

$$\varphi = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}, \quad \psi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \beta \\ 0 & 0 & 0 & \alpha \end{pmatrix},$$

and we have

$$(5.1) \quad \begin{aligned} B_{1j} \circ \varphi &= \delta(aB_{1j}+bB_{2j}), \quad B_{i3} \circ \psi = B_{i3} + \beta B_{i4}, \\ B_{2j} \circ \varphi &= \delta(cB_{1j}+dB_{2j}), \quad B_{i4} \circ \psi = \alpha B_{i4}, \qquad i = 1,2, \quad j = 3,4. \end{aligned}$$

<u>Theorem</u>. There are 7 orbits of $\mathrm{Aut}(\mathbb{R} \times \mathcal{G}_3)$ in $G_2(H^2(\mathbb{R} \times \mathcal{G}_3))$. We write them as $\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_3,$ Q_1, Q_3, Q_4, and Q_4^c where the subscript is the dimension of the orbit, and

$$\begin{aligned} \{(B_{14},B_{24})\} &= \mathcal{O}_0 \\ (B_{14},B_{13}) &\in \mathcal{O}_1 \\ (B_{14},B_{23}) &\in \mathcal{O}_3 \\ (B_{13},B_{23}) &\in Q_1 \\ (B_{13},B_{23}+B_{14}) &\in Q_3 \\ (B_{13},B_{23}+B_{24}) &\in Q_4 \\ (B_{13}+B_{24},B_{23}-B_{14}) &\in Q_4^c \end{aligned}$$

<u>Proof</u>. It follows from (5.1) that $P_4 = (B_{14},B_{24})$ is $\mathrm{Aut}(\mathcal{G})$-invariant. Let X be the closed subspace of $G_2(H^2(\mathcal{G}))$ consisting of the planes P with $P \cap P_4 \neq 0$. It is not difficult to see that X is the disjoint union of the following three sets,

$$\mathcal{O}_0 = \{P_4\}$$
$$\mathcal{O}_1 = \{(sB_{14}+tB_{24}, sB_{13}+tB_{23}) : (s,t) \neq (0,0)\}$$
$$\mathcal{O}_3 = \{(sB_{14}+tB_{24}, u(tB_{14}-sB_{24})+s_1B_{13}+t_1B_{23}) : (s,t) \neq (0,0), (s_1,t_1) \neq (0,0), (u, st_1-ts_1) \neq (0,0)\}$$

By directly computing the orbit through (B_{14},B_{13}) and (B_{14},B_{23}), using formulas (5.1), we find that they are $\mathcal{O}_1$ and $\mathcal{O}_3$. Clearly $\dim \mathcal{O}_1 = 1$, and $\dim \mathcal{O}_3 = \dim X = 3$.
Next we have to consider the planes P with $P \cap P_4 = (0)$. Let $\pi : H^2(\mathcal{G}) \to (B_{13},B_{23})$ be the projection with kernel P_4. Then $\pi(P) = (B_{13},B_{23})$ and hence $P = (B_{13}+v_1, B_{23}+v_2)$ where $v_1, v_2 \in P_4$ are uniquely determined by P. Setting

$$v_1 = a_{11}B_{14} + a_{21}B_{24}, \qquad A(P) = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix},$$
$$v_2 = a_{12}B_{14} + a_{22}B_{24},$$

the matrix $A(P)$ is uniquely determined by P, and every 2 by 2 matrix A is of the form $A(P)$ for a unique P. Setting $A(P)\varphi = A(P\circ\varphi)$ and $A(P)\psi = A(P\circ\psi)$, a direct computation, using (5.1) shows that, for any A,

$$A\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} A \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad A\psi = \alpha A + \beta I.$$

This defines an action of $\mathrm{Aut}(\mathcal{G})$ in the affine space of 2 by 2 matrices, and it is not difficult to see that the orbits are

Q_1 : scalar matrices,

Q_3 : matrices with only one eigenvalue, but which are not scalar,

Q_4 : matrices with two real eigenvalues, and

Q_4^c : matrices with complex eigenvalues.

Here Q_4 and Q_4^c are open orbits. Also, $\dim Q_3 = 3$ because $Q_3 \cup Q_1$ is the variety in $\mathbb{R}^4$ defined by $(a_{11}-a_{22})^2 = 4a_{12}a_{21}$.

Choosing representatives for the orbits such as $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we obtain representatives for the corresponding orbits in $G_2(H^2(\mathcal{G}))$,

$$(B_{13}, B_{23}) \in Q_1, \quad (B_{13}, B_{23}+B_{14}) \in Q_3$$

$$(B_{13}, B_{23}+B_{24}) \in Q_4, \quad (B_{13}+B_{24}, B_{23}-B_{14}) \in Q_4^c .$$

This completes the proof of the theorem. We note that $\mathcal{O}_0$ and Q_1 are degenerate orbits in the sense that $\mathfrak{z}_{P_4} \cap \mathfrak{z} = (e_3) \neq (0)$ and $\mathfrak{z}_{B_{13}} \cap \mathfrak{z}_{B_{23}} \cap \mathfrak{z} = (e_4) \neq (0)$. Hence, there are 5 Lie algebras with center of dimension 2 obtained from the remaining 5 orbits.

<u>5.6. Table</u>. Six-dimensional real nilpotent Lie algebras $\tilde{\mathcal{G}}$ with center $\tilde{\mathfrak{z}}$ of dimension ≥ 2. $\mathcal{G} = \tilde{\mathcal{G}}/\tilde{\mathfrak{z}}$.

$\mathcal{G}$	Defining orbit in $G_k(H^2(\mathcal{G}))$; $k = 2,3$	Product in $\tilde{\mathcal{G}}$ (only non-zero brackets are given).	$\tilde{\mathcal{G}}$
$\mathbb{R}^3$	$k = 3$ $\{H^2(\mathcal{G})\} = \{\Lambda_2\mathbb{R}^3\}$	$[e_1,e_2] = e_4$ $[e_2,e_3] = e_5$ $[e_1,e_3] = e_6$	$\mathcal{G}_{6,14}$
$\mathbb{R}^4$	$k = 2$ Ω_2	$[e_1,e_2] = e_5$ $[e_3,e_4] = e_6$	$\mathcal{G}_3 \times \mathcal{G}_3$
	Ω_3	$[e_1,e_2] = e_5$ $[e_1,e_3] = [e_2,e_4] = e_6$	$\mathcal{G}_{6,15}$
	Ω_4	$[e_1,e_2] = [e_3,e_4] = e_5$ $[e_2,e_3] = [e_1,e_4] = e_6$	$\mathcal{G}_{6,16}$
$\mathbb{R} \times \mathcal{G}_3$	$k = 2$ $\mathcal{O}_1$	$[e_1,e_2] = e_3$ $[e_1,e_3] = e_5$ $[e_1,e_4] = e_6$	$\mathcal{G}_{6,17}$
	$\mathcal{O}_3$	$[e_1,e_2] = e_3$ $[e_2,e_3] = e_5$ $[e_1,e_4] = e_6$	$\mathcal{G}_{6,18}$

	Q_3	$[e_1,e_2]=e_3$ $[e_1,e_3]=e_5$ $[e_1,e_4]=[e_2,e_3]=e_6$	$\mathcal{G}_{6,19}$
	Q_4	$[e_1,e_2]=e_3$ $[e_1,e_3]=e_5$ $[e_2,e_3]=[e_2,e_4]=e_6$	$\mathcal{G}_{6,20}$
	Q_4^c	$[e_1,e_2]=e_3$ $[e_1,e_3]=[e_2,e_4]=e_5$ $[e_2,e_3]=-[e_1,e_4]=e_6$	$\mathcal{G}_{6,21}$
$\mathcal{G}_4$	$k=2$ $\{H^2(\mathcal{G})\}=$ $\{(B_{14},B_{23})\}$	$[e_1,e_2]=e_3$ $[e_1,e_3]=e_4$ $[e_1,e_4]=e_5$ $[e_2,e_3]=e_6$	$\mathcal{G}_{6,22}$

5.7. Clearly the real nilpotent Lie algebras of dimension six containing direct factors are, within isomorphisms,

$$\mathcal{G}_1^6 \;;\; \mathcal{G}_3\times\mathcal{G}_1^3 \;;\; \mathcal{G}_4\times\mathcal{G}_1^2 \;;\; \mathcal{G}_{5,i}\times\mathcal{G}_1\;,\; 1\leq i\leq 6 \;;\; \mathcal{G}_3\times\mathcal{G}_3 \,.$$

Theorem. Every six-dimensional real nilpotent Lie algebra with no nontrivial direct factor is isomorphic to one of the following algebras

$$\mathcal{G}_{6,1},\mathcal{G}_{6,2},\ldots,\mathcal{G}_{6,22}\,.$$

These algebras are pairwise nonisomorphic.

Reference

J. Dixmier, Sur les représentations unitaires des groupes de Lie nilpotents, III, Canadian Journal of mathematics, 10 (1958) 321-348.